WEIBULL PROBABILISTIC APPROACH TO THE STRENGTH OF COMPOSITE MATERIALS OR COMPLEX STRUCTURES

P. Kittl*, G. Díaz and V. Martínez**

* Departamento de Ingeniería Mecánica,
Facultad de Ciencias Físicas y Matemáticas,
Universidad de Chile, Casilla 2777, Santiago, Chile

** Departamento de Ciencia de los Materiales,
Facultad de Ciencias Físicas y Matemáticas,
Universidad de Chile, Casilla 2777, Santiago, Chile

ABSTRACT

A summary concerning the basic hypothesis of the Weibull's theory of the probabilistic strength of a generalized composite material or complex structure is given. It is also show how this theory can be applied in order to find the ultimate strength. One result from the analysis is that the model for the fracture or plastic deformation of a composite material is determined by a series of two random numbers, one determining the stress of rupture or the transition from linear to non-linear deformation and the other the zone where it is produced. A general structure of both brittle and ductile members requires a series of eight random numbers. When having and extremely complex structure, as the earth crust, it can be affirmed that the location and magnitude of a fracture (earthquake) is impossible to predict.

INTRODUCTION

In order to develop a theory for composite materials, many works including deterministic and stochastic approaches [1-3] have been formulated. However, until today there are no works employing the probabilistic strength of materials [4] in a generalized form. Using the theory of probabilistic strength of materials, the authors of the present work have treated the case of a composite formed by a ductile fibre embedded in a brittle matrix studying the pull-out in a generalized form [5] and the debonding [6]. Experimental works were directed to copper fibres embedded in a cement matrix [7-14].

In a composite material or in a complex structure of brittle materials, the process of fracture is initiated by the propagation of the first crack and continues with the stress redistribution having many complexities. In general, only those cases in which, after the successive breakings, the stress field can be determined as a simple form have been treated [15]. If the system is not brittle, or if it is only partially so, the problem becomes complex again. In this case, only de simple instances have been treated too [16]. For example, the simplest case of a composite system is the case of a system of parallel brittle bars, and it was partially studied by Kittl and Díaz [17] and later by Martínez [18] in a generalized form. On the other hand, if the composite system consists only of ductile elements, σ_f being the stress when the plastic determination begins, the event can be treated like a problem where the elements are fractured just at the stress σ_f. After reaching σ_f, an increase of deformation does not mean an increase of tension. On other words, the material does not undergo work-hardening, or it is an elastic-perfectly plastic material. Moreover, assuming plastic deformation of a given ductile element after reaching the stress σ_f, then it is possible to suppose that this element no longer works when the material reaches the deformation corresponding to the maximal necking. When this happens the respective element can be removed from the analysis, as can be done with a brittle element.

The other problem is that, normally, one deals with composite materials formed by a matrix containing a disperse phase, as for example filaments, which have a random distribution and, hence, they have to be considered from the statistical standpoint.

The aim for this work is to provide a general approach which describes the behaviour relative to the fracture or plastic deformation of all the composite materials or complex structures using probabilistic strength of materials that considers randomness of the mechanical properties of every one of its constituents.

BASIC HYPOTHESIS

Let a composite or complex structure consisting of several materials be hypothetically subdivided into series elements, which are destroyed with the first fracture, and parallel elements, which survive after the first fracture when the stress field is applied. Taking into account a basic composite material constituted by series elements arranged in parallel, formed by K volumes, M joinings (boundaries) and P surfaces subjected to some stress field, as shown in Figure 1, it is possible to obtain, by using probabilistic strength of materials, an expression for the probability of the first crack or first fracture which would eliminate any volume, boundary or joining when the crack is produced. This treatment assumes that the system is brittle and the stress is gradually applied. When the first crack is propagated then the stress field is changed, and thus, a second fracture is produced in another series element of the composite, and so on until all the composite is broken. The relation between K, M and P is arbitrary as deduced by observing Figure 1, where a generalized composite is formed by volumes V_i, interfaces S_{ij} and free surfaces S_{i0}, and by arranging them in series and in parallel.

Thus, according to the above hypothesis, the general equation of cumulative probability of fracture or yielding [4,17,19-20], F {σ}, of a composite material which corresponds to a body composed of N different materials distributed in V_i, volumes, S_{ij} interfaces and S_{i0} external surfaces, subjected to some uniaxial stress field σ (r) ≤ σ, is:

$$
\begin{aligned}
&F(\sigma)=1-\exp\left(-\xi(\sigma)\right);\sigma_n=\sigma_n(\sigma);\tau=\tau(\sigma)\\
&\xi(\sigma)=\frac{1}{V_0}\sum_i\int_{V_i}\phi_i^V(\sigma(r))dV_i+\frac{1}{S_o}\sum_{i,j}\int_{S_{ij}=V_i\cap V_j\neq 0}\phi_{ij}^{\tau}(\tau(r))dS_{ij}\\
&+\frac{1}{S_o}\sum_{i,j}\int_{S_{ij}=V_i\cap V_j\neq 0}\phi_{ij}^{n}(\sigma_n(r))dS_{ij}+\frac{1}{S_o}\sum_i\int_{S_{i0}}\phi_{i0}^{n}(\sigma_n(r))dS_{i0}\\
&+\frac{1}{S_o}\sum_i\int_{S_{i0}}\phi_{i0}^{\tau}(\tau(r))dS_{i0}\qquad ;\quad i,j,=1,2,................N
\end{aligned}
\tag{1}
$$

In Equation (1) ϕ_i^V is the specific risk of volume fracture function for the material i, ϕ_{ij}^{τ} is the specific risk of surface fracture function of the interface between the material i and the material j subjected to a tangential stress τ, ϕ_{ij}^{n} is the specific risk of surface fracture function of the interface ij subjected to a normal stress σ_n, which is function of the maximum stress σ, ϕ_{i0}^{n} is the specific risk of surface fracture function of the external surface S_{i0} subjected to a normal stress σ_n, ϕ_{i0}^{τ} is the specific risk of surface fracture function of the external surface S_{i0} subjected to a tangential stress τ, which also is function of the maximal stress σ, r is the position vector, V_0 is the unit volume and S_0 is the unit surface. Equation (1) can be easily extended to the case of a triaxial stress field, as was deal with in other works [17,20].

The specific risk of volume or surface fracture or yielding function can be expressed in the form of a Weibull‘s function of three-parameter, as follows:

$$
\phi(\sigma)=\begin{cases}\left(\dfrac{\sigma-\sigma_L}{\sigma_0}\right)^m & \sigma\geq\sigma_L\\ 0 & \sigma<\sigma_L\end{cases}
\tag{2}
$$

where m and σ_o are the Weibull parameters depending on the manufacturing process of the material and σ_L is the lower limit stress under which there is no fracture. When the σ_L parameter is zero then equation (2) is transformed into a two-parameter Weibull function. Another expression of this specific risk of fracture function is the one of Kies-Kittl:

$$\phi(\sigma)=\begin{cases}\left(\dfrac{\sigma-\sigma_L}{\sigma_0}\right)^m & \sigma_L\le\sigma\le\sigma_S\\ 0 & \sigma<\sigma_L\\ \infty & \sigma>\sigma_S\end{cases} \tag{3}$$

where σ_S is the superior limit stress over which there is always fracture, this is because all materials have a minimal flaw size, and K is the Kittl´s constant. σ_L and σ_S proceed from the corresponding maximum and minimum cracks size.

The criterion of selection for the expression of the function $\phi(\sigma)$, given in equations (2) and (3), is essentially determined by experimental results. For example, Weibull [4], in his first paper, experimentally studied a great number of materials and concluded that the function giving the best fit with the data was a potential function of the expression given by equation (2). Later, by testing other materials, for example glass, the best fit was given by the function $\phi(\sigma)$ of equation (3), as was obtained by Kies [21] and applied later by Díaz and Morales [22].

SCALE DEPENDENCE

From formula (1), a dependence between the cumulative probability of fracture or yielding and the volume of the composite or complex structure can be obsereved. This was shown before in other works for simpler systems [4, 17, 19, 22]. This can be clearly seen by considering a uniform stress field, i.e., σ (r) = σ. Thus, from equation (1), when i=1 and the free surface S_{10} is not considered, one obtains:

$$F(\sigma,V)=1-\exp\left(-\frac{V}{V_0}\phi(\sigma)\right) \tag{4}$$

The above equation (4) allows us to determine the change of the function of cumulative probability of fracture when, in this case, the volume of the material is changed.

In many cases the behaviour, due to the change of volume, does not vary according to this theoretical functional dependence with the volume. However, this result can be obtained by using only the topological properties of the function F (σ, V) [19]. So, if we accept that

$$
\begin{aligned}
&F(\sigma,V) \quad \text{is uniform} \\
&F(0,V) = F(\sigma,0) = 0 \\
&F(\infty,0) = F(0,\infty) = F(\infty,\infty) = 1 \\
&F(\sigma,V) \geq 0 \text{ differentiable and crescent}
\end{aligned} \tag{5}
$$

then the existence of an unique function satisfying the above properties (5) can be found:

$$
\begin{aligned}
&F(\sigma,V) = 1 - \exp\left\{-g\left(\frac{V}{V_0}\right)\phi(\sigma)\right\} \\
&0 \leq g\left(\frac{V}{V_0}\right), \phi(\sigma) \leq \infty \text{ differentiable and crescent} \\
&\mathrm{g}(0) = \phi(0) = 0
\end{aligned} \tag{6}
$$

More details can be found in other works [19,20]. A model of a composite material can be constructed only when $g(V/V_o)$ is linear. In the opposite case, the model is not unique and depends on the subdivision applied to the volume. Thus, taking into account the case when $g(V/V_o)$ is linear, the following is manifest:

$$
\begin{aligned}
&\tilde{F}(\sigma,V) = 1 - F(\sigma,V) \\
&\tilde{F}(\sigma,V_1+V_2) = \tilde{F}(\sigma,V_1)\tilde{F}(\sigma,V_2)
\end{aligned} \tag{7}
$$

where $\tilde{F}$ is the cumulative probability of non fracture or non yielding. When properties (7) is satisfied, it can be established that formula (1) describes the behaviour of a generalized composite material prior to its first fracture. Formula (1) shows how the cumulative probability of fracture varies with the stress. If x is a random number, with $0 \leq x \leq 1$, by equating F (σ,V)=x, the stress σ at which the first fracture is produced with probability F(σ,V) in at least one of the components of the series composite material can be found. Therefore, the set $\{x_i\}$, x_i being random, is a realization from the model of the behaviour of the composite material for the first fracture, and this can be made by establishing a correspondence between every random number x_i and every component of the composite material (Montecarlo simulation). It is necessary to mention that property (7), wherefrom formula (1) is deduced, does not describe the probability of propagation of a crack but the probability that it does not propagate. So, the probabilities of fracture or plastic deformation and of non-fracture or non-plastic deformation are connected. From now on, the fracture will be indistinctly referred to as the fracture or the yielding. It is a common mistake to affirm that Weibull's formula, whose generalization is the formula (1), obtained from $F(\sum(\sigma_i,V_i)=\prod F(\sigma_i,V_i)$, is deduced by accepting that a microcrack is initiated at the weakest defect in the specimen. But, for obtaining formula (1) this hypothesis is not used, neither explicitly nor implicitly. The assumption employed is the fact that the cumulative probability of non-propagation of the crack obeys a similar law as the product of independent probabilities as shown in equation (7).

It is worthwhile to mention that, when a complex structure is either brittle or ductile, then it is possible to find, from the statistical standpoint, a connection between the fracture and the plastic deformation of one element, and between the ultimate strength and the total plastic deformation of the whole structure, respectively.

LOCAL PROBABILITY AND LOCATION OF THE FIRST FRACTURE

For the case of composite materials, the location where the first fracture appears is very important because this fracture propagates until one or more of the composite parts are broken or even until the system reaches the total collapse. In every stage, after a given crack propagates, and assuming that the applied stress is being gradually increased, the stress field is changed and formula (1) has to be successively applied until, finally, the composite material no longer resists. As was mentioned above, this can easily be done in the simplest cases. For example, a system of parallel bars was solved partially by Kittl and Díaz [17] and completely by Martínez [18], where the composite is considered to be formed by only one phase which is subdivided and which can be assimilated to a rope, the oldest composite. However, in the general case the treatment is quite complex. In spite of the complexities, the determination of the location of the fracture is very important, and there exists a controversy about the local probability of fracture. It will be treated in detail in this work, first, for a simple body, where i=1 and $\phi_{10} = 0$ are adopted.

After oh and Finnie`s approach [23], in a body of volume V, the probability of fracture of a small volume dV located at the position vector r is derived according to the following steps. First, when the maximal stress is increased from σ to σ+dσ, it is assumed that $(1/V_0)\phi(\sigma f(r))$ is the proportion of fractures, dn/n, beginning in the small volume dV of V per unit volume, where dn is the number of fractures starting in dV and n is the total number of fractures in the body of volume V. Second, with the above supposition, the number of breaks per unit volume when the stress is increased from σ to σ+dσ, when the unit volume is assumed to be infinitely small, is obtained by:

$$\frac{\partial}{\partial \sigma}\left(\frac{1}{V_0}\phi\left[\sigma f(r)\right]\right)d\sigma \tag{8}$$

Third, the above expression (8) is multiplied by the probability of nonfracture in the remaining volume V-dV, which is $\tilde{F}(\sigma, V - dV)$ and is the same as $\tilde{F}(\sigma, V)$ because V-dV tends towards V (dV<<V). Thus one has:

$$\left[1 - F(\sigma, V)\right]\frac{\partial}{\partial \sigma}\left(\frac{1}{V_0}\phi\left[\sigma f(r)\right]\right) \tag{9}$$

being $\tilde{F}(\sigma, V) = 1 - F(\sigma, V)$. Finally, as this last result is treated as a frequency for all values of stress σ then expression (9) is integrated in all σ, which gives the relationship between the local probability of fracture and the specific risk of fracture function, given by Oh and Finnie in 1970 [23]:

$$\frac{1}{n}\frac{dn(r)}{dV}=\int_0^\infty \exp(-\frac{1}{V_0}\int_V \phi[\sigma f(r)]dV)\frac{\partial}{\partial\sigma}\left[\frac{\phi[\sigma f(r)]}{V_0}\right]d\sigma \tag{10}$$

where dn(r) is the number of fractures starting within a small volume dV located at the position vector r and n is the number of fractures in the whole body of volume V at a stress less than or equal to σ when N tests are carried out. The expression $(1/n)dn(r)/dV$ represents the percentage of fractures per unit volume starting within volume dV at a stress less than or equal to σ.

Equation (10) is defined for a volume dV which can be used to extend it to some volume $V' \leq V$, which is the procedure used to ascertain the local probability of fracture. Then Eq (10) becomes, according to Oh and Finnie's approach:

$$\frac{n(V')}{n}=\int_{V'}\left\{\int_0^\infty \exp\left[-\frac{1}{V_0}\int_V \phi[\sigma f(r)]dV\right]\frac{\partial}{\partial\sigma}\left[\frac{\phi[\sigma f(r)]}{V_0}\right]d\sigma\right\}dV' \tag{11}$$

Here it is worth observing that the integration is to be carried out first with respect to σ and thereafter in volume $V' \leq V$,

Now, let us see the local probability after Kittl and Camilo. By differentiation of Eq (1) for i=1 and $\phi_{10}=0$, one obtains:

$$\frac{dF}{1-F}=\phi[\sigma f(r)]\frac{dV}{V_0} \tag{12}$$

The expression:

$$\frac{dF}{F}=\frac{1-F}{F}\phi[\sigma f(r)]\frac{dV}{V_0} \tag{13}$$

represents the cumulative probability of failure in the volume dV, located at the position vector r. By integrating Eq (13):

$$\int_V \frac{dF(r)}{F(V)}=\frac{1}{F(V)}\int_V \frac{dF(r)}{dV}dV=\frac{F}{F}=1=\frac{1-F}{F}\int_V \phi[\sigma f(r)]\frac{dV}{V_0} \tag{14}$$

and by eliminating (1-F)/F between (13) and (14) we obtain the relationship between local probability of fracture and specific risk of fracture function, given by Kittl and Camilo in 1981 [24]:

$$\frac{1}{n}\frac{dn(r)}{dV}=\frac{\phi\left[\sigma f(r)\right]}{\int_{v}\left[\sigma f(r)\right]dV} \tag{15}$$

where n is the total number of failed bodies and dn(r) are those which began to fail within the volume dV located in r. Equation (15) gives the fraction of failures per unit volume at the point r of the body at the stress σ.

It is worth remembering that the integration of (14) and (15) is over all values of $r\in V$, that is to say, the variable is r. If the percentage of fractures existing in the volume $V_1 \leq V$ is required, then Eq (15) has to be integrated over the volume V_1.

Equation (10) has been used by Schultrich and Fährmann [25] for determining the relationship between the local probability of fracture and variance in the three-point bending test of rectangular beams. Trustum [26] used Eq (10) in order to assess Weibull`s parameter m employing the local probability of fracture and the fracture stress in the three-point bending test. Both works [25,26] used the method of defined functions with a two-parameter specific risk of fracture function, that is to say, Eq. (2) with $\sigma_L = 0$. Probably these authors have assumed that formula (10) is correct because of the following result:

$$\int_{r\in V}\frac{1}{n}\frac{dn(r)}{dV}dV=\frac{1}{n}\int_{r\in V}\frac{dn(r)}{dV}dV=\frac{n}{n}=1 \tag{16}$$

or, alternatively:

$$\begin{aligned}\int_{r\in V}\frac{1}{n}\frac{dn(r)}{dv} &= \int_{r\in V}\int_0^{\infty}\exp\left(-\frac{1}{V_0}\int_V\phi\left[\sigma f(r)\right]dV\right)\frac{\partial}{\partial\sigma}\left[\frac{\phi(\sigma f(r))}{V_0}\right]d\sigma dV \\ &=\int_0^{\infty}\exp\left(-\frac{1}{V_0}\int_V\phi\left[\sigma f(r)\right]dV\right)\frac{\partial}{\partial\sigma}\left(\frac{1}{V_0}\int_V\phi(\sigma f(r))dV\right)d\sigma \qquad ;r\in V \\ &=\int_0^{\infty}e^{-\xi}\frac{d\xi(\sigma)}{d\sigma}d\sigma=1\end{aligned} \tag{17}$$

where the function $\xi(\sigma)$, termed Evans` function [27], is given by:

$$\xi(\sigma)=1n\frac{1}{1-F}=\frac{1}{V_0}\int_V\phi\left[\sigma f(r)\right]dV \tag{18}$$

Now then, let us analyze in detail the steps of the Oh and Finnnie's formulation. For that, consider the following result corresponding to Weibull's probabilistic approach and which can be easily deduced from Eq (12):

$$\begin{aligned} &dF = \frac{dn(r)}{N} = (1-F)\frac{\phi(\sigma f(r)}{V_0}dV \\ &1-F = 1-\frac{n}{N} = \frac{N-n}{N} \\ &\rightarrow \frac{dn(r)}{N(1-F)} = \frac{dn(r)}{N-n} = \frac{dV}{V_o}\phi(\sigma f(r)) \end{aligned} \tag{19}$$

where N is the total number of failed bodies, n those failed bodies under at a stress less than or equal to a given stress σ and dn(r) those which began to fail within the volume dV located at r at stress less than or equal to σ. Consider now the first assumption made in the Oh and Finnie's approach, which can be written as follows:

$$\frac{dn(r)}{n} = \frac{dV}{V_0}\phi\left[\sigma f(r)\right] \tag{20}$$

where dn(r) /n is the percentage of fractures beginning within volume dV at the point located at r when the maximal stress is increased from σ to σ + d σ. Finally, consider the cumulative probability of fracture of a small volume dV, which is easily obtained from Eq (1) for i=1 and $\phi_{10} = 0$, presented in accordance to the results of Eq (19) and (20):

$$\begin{aligned} &\frac{dV}{V_0}\phi\left[\sigma f(r)\right] = F(\sigma, dV) \\ &= \frac{dn(r)}{N-n} \quad \text{(Weibull)} \\ &= \frac{dn(r)}{n} \quad \text{(Oh and Finnie)} \end{aligned} \tag{21}$$

The following comments can be extracted from above deductions of Eqs (19) to (21). First, from Eq (21) and the result for dn(r)/n given by Oh and Finnie, it is not wholly evident of the existence of a relation which connects the probability $F(\sigma, dV)$ of an isolated small volume dV, neither with the total number of failed bodies of volume V, N, nor with the number of failed bodies that began fracture in the remaining part of the body, V-dV, at a stress less than or equal to σ. It must be remembered that every test has to be carried out by increasing the stress until fracture of the body. When this is done, it is clear that some of tests realized in the volume V-dV up to fracture do not initiate the fracture in the volume dV, which leads to a different relation for the percentage of fractures. This is the reason of the difference expressed in Eq (21), where this percentage, according to the Weibull's theory is given by dn(r)/N-n), as deduced in Eq (19), but in the theory of Oh and Finnie this would be given by dn(r)/n when the stress is increased from σ to σ+dσ. Second, to obtain the formula (10), which is deduced from formula (9), the probability of fracture in dV is multiplied by the probability of non-fracture in the remaining volume V-dV. If this were made to obtain the probability of fracture of a body subdivided in two disjoint volumes V_1 and V_2, so that $V_i+V_2=V$, $F(V_1)$, $F(V_2)$, $\tilde{F}(V_1)$ and $\tilde{F}(V_2)$ being respectively the probabilities of fracture and nonfracture of body of volume V_1 and V_2, one would have:

$$F(V_1+V_2)=F(V_1)\tilde{F}(V_2) \tag{22}$$

and likewise:

$$F(V_1+V_2)=\tilde{F}(V_1)F(V_2) \tag{23}$$

which evidently is false and invalidates the formulas (9) and (10). The accuracy that can be attributed to formula (10) resides in the fact that the two theories give the same result when the Weibull's function is one of two parameters, in other terms $\phi(\sigma)=(\sigma/\sigma_0)^m$. In effect, as in this case the function ϕ is decomposed as $\phi[\sigma(r)]=\phi[\sigma f(r)]=\phi(\sigma)\phi[f(r)]$ then Eq (10) for the local probability of fracture after Oh and Finnie gives:

$$\begin{aligned}\frac{dn(r)}{n}&=dV\int_0^\infty\left[-\frac{\phi(\sigma)}{V_0}\int_V\phi[f(r)]dV\right]\frac{1}{V_0}\phi[f(r)]\phi'\{\sigma\}d\sigma\\&=\frac{\phi[f(r)]dV}{\int_V\phi[f(r)]dV}\end{aligned} \tag{24}$$

and, as can be easily observed, the same result is obtained from Eq. (15). Consequently, in the case of some two parameter Weibullian specific risk function, when $\sigma_L=0$ in Eq (2), both expressions yield the same result and moreover they are independent of stress σ.

In short, two logical defects have been detected in the theory of Oh and Finnie. First, using the expression $(l/n)(dn(r)/dV)$ instead of $(1/(N-n))(dn(r)/dV)$ taking into account that $F(\sigma, V)=n/N$ where n is the number of fractures in V during the test $N\geq n$ at a stress less than or equal to σ and, second, multiplying by $1-F(\sigma,V)=F(\sigma,V)$ because, evidently, there is not any brittle body fracturing in V_1, and surviving in V_2 if V_1 and V_2 are constituting the same body of volume $V = V_1 + V_2$. These two mentioned errors are compensated and the results of formulas (16), (17) and (24) are obtained. The formulas (16) and (17) are coherent and the formula (24) is true, but the general formula (10) is mistaken.

Now the instance of $\sigma_L \neq 0$, that is to say a three-parameter function $\phi(\sigma)$, will be considered. Hence, consider a rectangular beam L long, h high and b wide, subjected to the three-point bending test with a fracture load P. Then the stress field is given by:

$$\begin{gathered}\sigma(r)=\sigma(x,y,z)=\frac{4xy}{Lh}\sigma \leq \sigma=\frac{3}{2}\frac{PL}{bh^2}\\ 0\leq x\leq L/2; \leq y\leq h/2; -b/2\leq z\geq b/2\end{gathered} \tag{25}$$

If $\sigma_L = 0$ in Eq (2) then Eqs (10) and (15) give:

$$\frac{\mathrm{n(x)}}{\mathrm{n}}=\left(\frac{2x}{L}\right)^{m+1} \tag{26}$$

which result is independent of σ and takes, obviously, the value 1 when x=L/2. If considering Eq (25) and Eq (2) with $\sigma_L \neq 0$ then Eq (10) for the local probability of fracture after Oh and Finnie becomes:

$$\begin{aligned}\frac{n(x)}{n}=&\frac{bhL}{2V_0(m+1)}\left(\frac{\sigma_L}{\sigma_0}\right)^m \sigma_L\int_{\sigma L}^{\infty}\exp\Big\{\left(-\frac{bhL}{2V_0(m+1)}\right)\left(\frac{\sigma_L}{\sigma_0}\right)^m\\ &\times\frac{1}{\sigma/\sigma_L}\int_1^{\sigma/\sigma_L}\frac{(\eta-1)^{m+1}}{\eta}d\eta\Big\}\frac{1}{\sigma^2}\left[\frac{m}{m+1}\left(\frac{2x\sigma}{L\sigma_L}-1\right)^{m+1}+\int_1^{\frac{2x}{L}\frac{\sigma}{\sigma_L}}\left(\frac{(\eta-1)^m}{\eta}\right)d\eta\right]d\sigma\end{aligned} \tag{27}$$

This Eq (27) supplies the percentage of fractures for any position x and it is not equal to 1 when x=L/2. Hence, Eq (10) is inconsistent in this case and, therefore, Oh and Finnie's formula (10) is a case in which, in a peculiar form, and example of a logic rule is produced; in other words, one coherent expression is not necessarily true in theory.

On the other hand, the theory of Kittl and Camilo has the advantage that formula (15) can be obtained by another way, independent of the one already followed above. Let F_L be the local probability of fracture or the percentage of fractures starting in a given region of the body subjected to some stress σ. Consider also a disjointed and arbitrary division of an homogeneous body of volume V into volumes V_1 and V_2 . If n_1 and n_2 are the number of fractures that are started respectively in volumes V_1 and V_2, $N = n_1 + n_2$ being the total number of tests carried out — with $n_1/n_2=V_1/V_2$ [24] —, then the percentages of fractures that are started in V_1 and V_2, respectively, are:

$$\begin{gathered}F_L(\sigma,V_1)=\frac{n_1}{N}\\ F_L(\sigma,V_2)=\frac{n_2}{N}\end{gathered} \tag{28}$$

Therefore, the local probability of fracture is given by the following functional equation [20,28]:

$$F_L\left(\sigma,V_1\right)V_2 - F_L(\sigma,V_2)V_1 = 0 \tag{29}$$

which means that the local cumulative probability, $F_L(V)$, for the fracture to begin in the volume V, is proportional to it. Putting V_1=V and V_2=V + $\Delta V, \Delta V \to 0$, yields:

$$F_L(\sigma,V) = \frac{dF_L(\sigma,V)}{dV}V = F_L'\left(\sigma,V\right)V \tag{30}$$

whose solution is:

$$0 \le F_L(\sigma,V) = \frac{V}{V_0}\phi_L\left(\sigma\right) \le 1 \tag{31}$$

in which $\phi_L(\sigma)$ must be determined.

The above functional equation (29) ‒ whose solution is Eq (31) ‒ represents one of the principles of Probabilistic Strength of Materials [20], and it is independent and incompatible with the functional equation of the total cumulative probability of fracture except in the region where $(V/V_0)\,\phi_L(\sigma)$ tends towards zero. Hence, in that boundary the function ϕ_L must be equal to the specific risk function of the total cumulative probability of fracture, i.e. $\phi_L = \phi,$ and the following is satisfied:

$$\begin{aligned}
&F(\sigma,V) = F_L\left(\sigma,V\right) = \left(\frac{V}{V_0}\right)\phi\left(\sigma\right) \\
&\frac{V}{V_0}\phi(\sigma) \to \frac{dV}{V_0}\phi(\sigma) \to 0 \\
&F_L \to \frac{dn}{N}
\end{aligned} \tag{32}$$

From last Eq (32), the following result is obtained:

$$\frac{dn}{N} = \frac{\phi\left[\sigma(r)\right]}{V_0}dV \tag{33}$$

where dn is the number of fractures started within volume dV located at the position vector r. The integration of Eq (33) produces:

$$\frac{n}{N}=\frac{1}{V_0}\int_V \phi\left[\sigma(r)\right]dV \qquad ;r\in V\left[\sigma(r)\geq 0\right] \tag{34}$$

and the elimination of N from Eqs (33) and (34) results again in Eq (15) of the local probability after Kittl and Camilo. Hence, the formula (15) has been deduced of two independent and different forms, which insures that its formulation is consistent.

If the stress-space [17] is used, the physical space is subdivided into zones, as presented in the following equations:

$$\overline{r}\in \Delta V(\Sigma); \qquad 0<\Sigma\leq\sigma(r)\leq\Sigma+\Delta\Sigma<\sigma \tag{35}$$

Considering the above subdivision, the expression (1) is transformed into:

$$\begin{aligned}
\xi(\sigma) &= \frac{1}{V_0}\sum_i\int_0^{\sigma}\phi_i^V(\Sigma)S\left(\frac{\Sigma}{\sigma}\right)d\Sigma+\frac{1}{S_0}\sum_{\substack{i,j\\ S_{ij}=V_i\cap V_j\neq 0}}\int_0^{\tau(\sigma)}\phi_{ij}^{\tau}(T)L\left(\frac{T}{\tau}\right)dT\\
&+\frac{1}{S_0}\sum_{\substack{i,j\\ S_{ij}=V_i\cap V_j\neq 0}}\int_0^{\sigma_n(\sigma)}\phi_{ij}^{n}(\Sigma_n)L\left(\frac{\Sigma_n}{\sigma_n}\right)d\Sigma_n+\frac{1}{S_0}\sum_i\int_0^{\sigma_n(\sigma)}\phi_{io}^{n}(\Sigma_n)L\left(\frac{\Sigma_n}{\sigma_n}\right)d\Sigma_n\\
&+\frac{1}{S_0}\sum_i\int_0^{\tau(\sigma)}\phi_{i0}^{\tau}(T)L\left(\frac{T}{\tau}\right)dT\\
F(\sigma,V) &= 1-\exp\left[-\xi(\sigma)\right]
\end{aligned} \tag{36}$$

where $0\leq\Sigma\leq\sigma$ is the uniaxial stress, $0\leq T\leq\tau$ is the tangential stress and $0\leq\Sigma_n\leq\sigma_n$ the normal stress, acting as is the volume V_Σ as in the interfaces S_{ij}, and S and L are functions of partition of the stress space and they naturally depend on the material geometry and on the loading.

MODELS OF FRACTURE IN THE SERIES AND PARALLEL COMPOSITE MATERIALS OR IN A COMPLEX STRUCTURE AND CONCLUSIONS

Formula (36) can express the process of total rupture of a composite material in a very clear form and can generate a model of the composite that can be numerically treated.

In a series composite material this process becomes determined by two random numbers, the first one determines the maximal stress σ reached at the fracture moment and the second one determines the location of the fracture. Therefore, if $0\leq x\leq 1$ it that said first random number then the equation $F(\sigma)=x$ gives us the maximal stress σ reached by the series composite material when this fractures. If $0\leq y\leq 1$ is the second random number then the equation $\Sigma/\sigma=y$ gives us the zone $\sigma(r)=\Sigma$ where the rupture is produced and after this the basics composite material in series is destroyed. On the other hand, a composite material in series is destroyed. On the other way, a composite material in parallel, which does not necessarily break when the first fracture appears, can be constructed by considering each component as formed by a series composite material. The cumulative probability of fracture in that system will be given by:

$$F(V,\sigma)=\prod_{i}^{N_p}\left[\tilde{F}_i(V_i,\sigma)+F_i(V_i,\sigma)\right]-\prod_{i}^{N_p}\tilde{F}(V_i,\sigma) \qquad (37)$$

where the product of addeds contains all of the probable cases where one or two or all of the basic systems in series are fractured minus the probability of non-fracture of anyone or all of them surviving [29]. In order to make a model of the parallel composite, N_p random numbers $(x_i)=(x_1,x_2,.....x_{Np})$ can be considered, which ones determine (σ_i) stresses of fracture $x_i=F_i(V_i,\sigma_i)(i=1,2,.......N_p)$. From that set, the minor value of the σ_i indicated us at what stress the first fracture is produced. This first fracture can have N_p-1 coincidences the interval at which the coincidence is produced is fixed, which is equivalent to the error at which the stress of rupture in the first test is ascertained. After this or these breakings, the stress field is changed and a new set of aleatory numbers have to be considered to solve the system changed by the first crack propagation. This process continues successively until the total fracture of the system is reached. Likewise, the places where the fracture is produced, in every one of the series systems, can be determined by another random number, y_i. The equation $y_i=\sum_j/\sigma_i$, gives us the location of the fracture at the stress σ_i. There must be remembered that stresses are going to gradually increase in this process and, after a fracture, an abrupt increase of the stress does not exist, the deformation must be gradually increased until to pass from the stress before the fracture to stress after the fracture. In the opposite way, an instantaneous increase of the stress must be considered, which implies multiplying them by two [17]. The treatment described here for a brittle material can be also applied for a ductile material, considering this as an elastic perfectly-plastic material, which allows us to replace the fracture by the beginning of plastic deformation.

On the other hand, if having a general complex structure containing simultaneously brittle and ductile components, then, four aleatory numbers, x_i ,y_j,z_k and u_m, can be defined instead of two ones as described in the above paragraph. The equation $x_i=F(V_i,\sigma_i)$ gives us the fracture stress of the brittle elements whereas $y^n{}_j=\sum^n{}_j/\sigma_j$ gives us the location of that fracture at the stress σ_i, n = (1,2,3). Likewise, for the ductile elements, the stress at the beginning of the plastic deformation is obtained from the equation $z_k=F(V_k,\sigma_k)$ whereas the equation $u_m=\sum_m/\sigma_m$ gives us the maximal necking reached before the fracture of the material.

The procedure above described allows us to analyse the most general composite material that can be constructed. This can be considered as a triangular or tetrahedral array connected by series systems, which ones can have several ones in parallel, and each one of these last ones can have, also, a viscoelastic and an elastic constant. A simulation method can be applied as in [29,30].

An example of great complexity is the earth crust, a natural and important general composite being able to be studied with equation (1). As the earth crust is a very complex structure and, likewise, has a complex stress field then, the predictions related to an earthquake such as the magnitude and epicenter are impossible to be made. The unique real possibility is to use a probabilistic model as the one shown here, because the magnitude of an earthquake is associated to the maximal stress at fracture and the epicentre is associated to the location of the earth crust fracture. Therefore, both the magnitude and the epicenter have a given probability being able to be ascertained by means of the probabilistic strength of materials applied to a complex structure like the earth crust (Kittl, Díaz y Martinez, 1993) [28].

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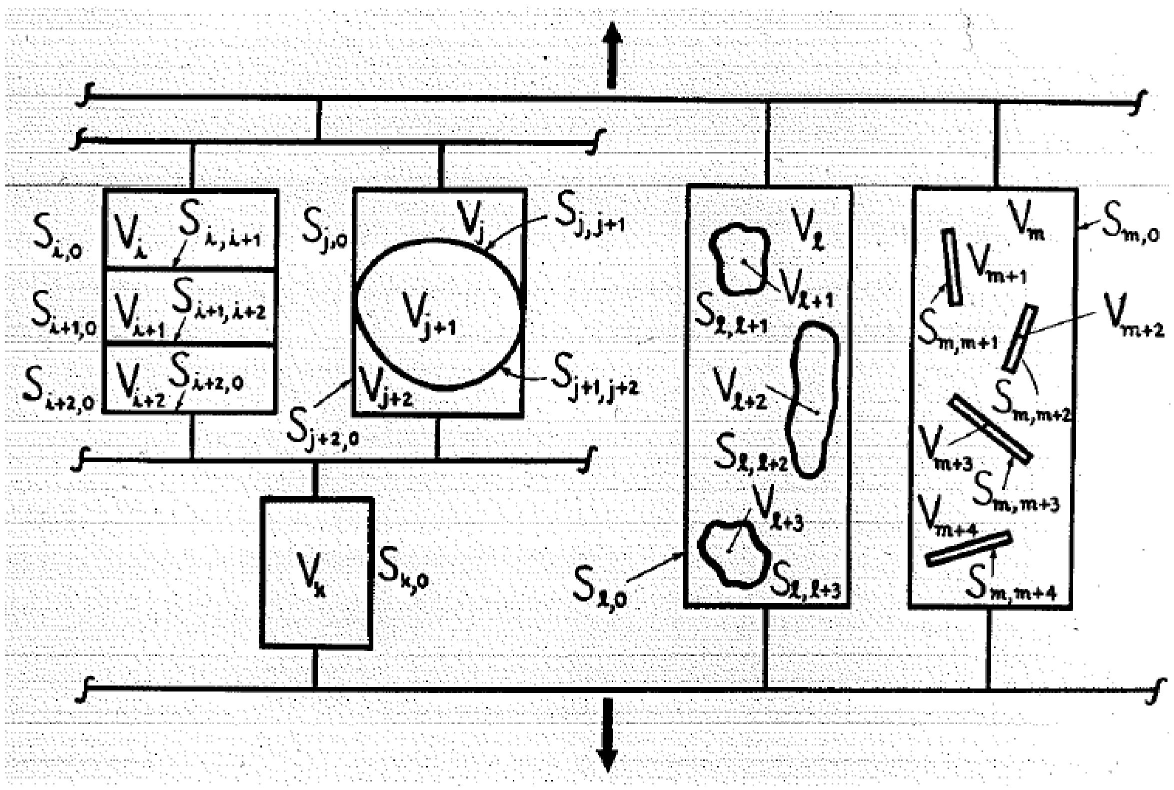

Figure 1: Generalized composite formed by volumes V_i, interfaces S_{ij} and free surfaces S_{i0}, and by arranging them in series and in parallel.